Some Aspects of Turbulent Relative Particle Dispersion

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Abstract

In this paper, several aspects of the turbulent relative particle dispersion are considered, many of which are motivated by laboratory experiment and numerical simulation results. These are,

* spatial intermittency effects,

  (a) reduction of relative particle dispersion in 3D;

  (b) prevalence of power-law scaling of relative particle dispersion in 2D enstrophy cascade;

* quasi-geostrophic aspects exhibiting an enhanced relative particle dispersion in the baroclinic regime and a negative eddy-viscosity development for some insight into this aspect;

* reduction of relative particle dispersion and prevalence of the ballistic regime due to compressibility effects.

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1. Introduction

Taylor [1] introduced the concept of diffusion by continuous movements and defined a diffusion coefficient D such that the mean square displacement $\langle [s(t)]^2 \rangle$ is given by

$$\langle [s(t)]^2 \rangle = 2Dt \tag{1}$$

t being the elapsed time. Richardson [2], on the other hand, proposed that turbulent diffusion should be characterized by the distance between neighboring particles, because if the inter-particle distance is within the inertial range, one may expect to find universal super-diffusive behavior in the relative particle dispersion process, which may be interpreted in terms of an interparticle separation dependent turbulent diffusivity[1]. From a purely empirical analysis of atmospheric dispersion data, Richardson [2] then showed that the turbulent diffusivity defined by the rate of increase of the mean square interparticle separation distance,

$$\mathcal{D} \equiv \frac{1}{2} \frac{d}{dt} \langle [R(t)]^2 \rangle \tag{2}$$

goes like $4/3$ power of this distance[2],

$$\mathcal{D} \sim \langle [R(t)]^2 \rangle^{2/3}. \tag{3}$$

The most important application of two-particle dispersion is in understanding the motion of passive scalars (like pollutants) in atmospheres and oceans.

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[1]Indeed, as Taylor [3] mentioned, "Richardson was a very interesting and original character who seldom thought on the same lines as his contemporaries and often was not understood by them."

[2]On theoretical grounds, (3) can be justified only if the advecting velocity field is finite correlated in time. This follows by noting, from the initial-value problem

$$\left. \begin{array}{l} \frac{d}{dt} R(t) = V(t) \\ t = 0 \; : \; R = R_0 \end{array} \right\}$$

that

$$\frac{1}{2} \langle [R(t) - R_0]^2 \rangle = \int_0^t \int_0^t \langle V(t')\, V(t'') \rangle \, dt'\, dt''.$$

Assuming a stationary process, this gives

$$\langle [R(t) - R_0]^2 \rangle = 2t \int_0^t C(\tau) d\tau$$

where $C(\tau)$ is the autocorrelation,

$$C(\tau) \equiv \langle V(0)V(\tau) \rangle.$$

If $C(\tau)$ decays in time fast enough so $\int_0^\infty C(\tau)d\tau$ is finite, we have a diffusive motion with turbulent diffusivity,

$$\mathcal{D} = \lim_{t \Rightarrow \infty} \frac{1}{2t} \langle [R(t) - R_0]^2 \rangle = \int_0^\infty C(\tau)d\tau.$$

On the other hand, if $C(\tau)$ decays slowly in time, we have a super-diffusive motion with

$$\langle [R(t) - R_0]^2 \rangle \sim t^{2\nu}, \; \nu > 1/2$$

and

$$\mathcal{D} \sim \langle [R(t) - R_0]^2 \rangle^{\frac{2\nu-1}{2\nu}}.$$

Obukhov [4] showed that Richardson's relation (3) can be derived via Kolmogorov's [5] (K41) theory[3] for homogeneous isotropic 3D fully developed turbulence (FDT). When the interparticle separation is within the inertial range, Obukhov [4] gave

$$\mathcal{D} \sim \epsilon^{1/3} \left\langle [R(t)]^2 \right\rangle^{2/3} \tag{4}$$

ϵ being the mean energy dissipation rate[4], hence the name Richardson-Obukhov (RO) scaling. It may be mentioned that Richardson [2] indicated that $\mathcal{D}$ has the dimensions of $\epsilon^{1/3}$ and that $\mathcal{D}$, like ϵ, remains nearly constant. So, Richardson [2], interestingly provided the first experimental evidence for dissipative anomaly (i.e., ϵ is almost independent of Reynolds number) and came very close to fully discovering the basic tenets of the Kolmogorov [5] theory 15 years earlier. On the other hand, the Richardson formulation connects with the universal aspects of FDT actually stronger than the Kolmogorov formulation since the unphysical effects due to sweeping by large scales are precluded from the outset in the Richardson formulation[5].

In this paper, several aspects of the turbulent relative particle dispersion are considered, many of which are motivated by laboratory experiment and numerical simulation results. These are,

* spatial intermittency effects,

  (a) reduction of relative particle dispersion in 3D, corroborating the numerical simulation results (Boffetta and Sokolov [7]);
  (b) prevalence of power law scaling of relative particle dispersion in 2D enstrophy cascade, corroborating the difficulty in observing Lin [8] scaling law in laboratory experiments (Jullien [9]);

* quasi-geostrophic aspects exhibiting some interesting departures from classical 2D results and a negative eddy-viscosity development for some insight into these aspects;

* particle clustering due to compressibility effects, corroborating the laboratory experiment and numerical simulation results (Cressman et al. [10]) where compressibility is effectively produced on the free surface of a shallow fluid layer.

2. 3D Relative Particle Dispersion

Consider the temporal evolution of the separation between two particle trajectories in the inertial regime $\eta \ll R \ll L$ in high R_e FDT. The equation for the separation distance between two particles is

$$\frac{d}{dt}\mathbf{R}(t) = \delta\mathbf{v}\left(\mathbf{R}, t\right) \tag{1}$$

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[3]The Kolmogorov [5] theory stipulates that the probability distribution function of velocity fluctuations, in the inertial range, depends only on the mean energy dissipation rate ϵ and the length scale ℓ. Further, ϵ is independent of the Reynolds number R_e.

[4]There is some logical inconsistency in the expression in (4) - $\mathcal{D}$ and $\left\langle [R(t)]^2 \right\rangle$ are Lagrangian quantities while ϵ is defined fully precisely in the Eulerian formulation.

[5]Indeed, as Monin and Yaglom [6] remarked, Richardson's formulation "indicates Richardson's belief in the existence of a universal physical law of sufficiently simple form."

from which,

$$\frac{d}{dt}[\mathbf{R}(t)]^2 = 2\mathbf{R}(t) \cdot \delta\mathbf{v}(\mathbf{R}, t) \tag{2}$$

Assuming the scaling relation,

$$|\delta\mathbf{v}(\mathbf{R}, t)| \sim [R(t)]^{\alpha}, \ \alpha < 1 \tag{3}$$

and averaging over many particles and many different initial separations, we obtain from (2),

$$\frac{d}{dt}\langle[R(t)]\rangle^2 \sim \langle[R(t)]^2\rangle^{(1+\alpha)/2} \tag{4}$$

from which,

$$\langle[R(t)]^2\rangle \sim t^{\frac{2}{1-\alpha}}. \tag{5}$$

Observe that (5) shows a super-diffusive growth since $\alpha < 1$ (for ordinary diffusion like the Brownian motion, $\langle[R(t)]^2\rangle \sim t$). For the K41 scaling given by $\alpha = 1/3$, (5) leads to the RO scaling,[6]

$$\langle[R(t)]^2\rangle \sim t^3. \tag{6}$$

3. 3D Relative Particle Dispersion: Effects of Spatial Intermittency

The RO theory does not take into account the spatial intermittency in FDT that was revealed by laboratory experiments and numerical simulations. As is well known, spatial intermittency effects would cause systematic departures from the RO scaling law (6) which uses mean energy transfer rate. On the other hand, spatial intermittency effects are known to become more pronounced at small scales, so the turbulence activity gets concentrated in smaller and smaller regions of space and the active region (called the dissipative structures) becomes strongly convoluted like a fractal (Mandelbrot [11]). Indeed, the smaller the particle separation, the stronger the spatial intermittency effects become. The fractal aspects of FDT may be simulated in a first approximation by representing the dissipative structures via a homogeneous fractal with non-integer Hansdorff dimension D (Frisch et al. [12]). Using then the result,

$$\alpha = \frac{1}{3} + \frac{D_0 - 3}{3} \tag{7}$$

(5) becomes,

$$\langle[R(t)]^2\rangle \sim t^{\left[3-3\left(\frac{3-D_0}{5-D_0}\right)\right]}. \tag{8}$$

Noting that $D_0 < 3$, (8) shows that the effect of spatial intermittency is to cause reduction in the relative dispersion, in agreement with the numerical results of Boffetta and Sokolov [7]. Further, (8) also shows that the spatial intermittency effects, no matter how strong

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[6]For a chaotic system (with positive Liapunov exponent λ), we have

$$R(t) \sim e^{\lambda t}.$$

In FDT, there is a unique $\delta v(R, t)$ for each R, so there is now a continuum of $\lambda(R)$. Consequently, $R(t)$ grows algebraically in FDT rather than exponentially.

(i.e. even in the limit $D_0 \Rightarrow 0$), cannot change the super-diffusive nature of relative particle dispersion, i.e.,

$$\left\langle [R(t)]^2 \right\rangle \sim t^\nu, \ \nu > 1. \tag{9}$$

It is of interest to note that the relative dispersion of particle pairs in FDT was also considered by Roberts [13] using the Eulerian DIA method (and eliminating the spurious effect of sweeping motion of the small eddies by the energy-containing eddies). If the energy spectrum is given by

$$E(k) \sim k^{-n} \tag{10}$$

Roberts [13] gave

$$\left\langle [R(t)]^2 \right\rangle \sim t^{\left(\frac{4}{3-n}\right)}. \tag{11}$$

In the presence of spatial intermittency, we have from (7),

$$n = \frac{5}{3} + \frac{1}{3} \left(3 - D_0 \right). \tag{12}$$

Using (12), (11) gives

$$\left\langle [R(t)]^2 \right\rangle \sim t^{\left[3 + 3 \left(\frac{3 - D_0}{D_0 + 1} \right) \right]} \tag{13}$$

which gives a relative particle dispersion faster than that given by the RO scaling law (6) contrary to the expected slow down of relative-particle dispersion by spatial intermittency.

4. 2D Relative Particle Dispersion

In 2D FDT, for pair separations larger than the energy injection scale but smaller than the integral scale (as in the inverse energy cascade), we have the scaling behavior,

$$\delta v(R) \sim R^{1/3} \tag{14}$$

which leads to the following growth relation,

$$\frac{d}{dt} \left\langle [R(t)]^2 \right\rangle \sim \left\langle [R(t)]^2 \right\rangle^{2/3} \tag{15}$$

and hence, the mean square pair separation grows as (Kowaleski and Peskin [14]),

$$\left\langle [R(t)]^2 \right\rangle \sim t^3. \tag{6}$$

On the other hand, when the pair separation exceeds the integral scale, the particles become uncorrelated and Brownian diffusion sets in with the following growth relation,

$$\left\langle [R(t)]^2 \right\rangle \sim t. \tag{16}$$

Okubo [15] analyzed oceanic experimental data and found the $R^{4/3}$ law for the relative diffusivity along with the t^3 law for the mean square pair separation, in agreement with (15) and (6), respectively. The laboratory experiments of Jullien et al. [16] also confirmed (6).

For pair separations much smaller than the energy-injection scale, the velocity field would be smooth. The relative particle dispersion in the 2D enstrophy cascade was considered by Lin [8] who gave for the relative diffusivity, the following scaling result,

$$\frac{d}{dt} \left\langle [R(t)]^2 \right\rangle \sim \tau^{1/3} \left\langle [R(t)]^2 \right\rangle \tag{17}$$

and hence, for the mean square separation, the exponential growth behavior,

$$\left\langle [R(t)]^2 \right\rangle \sim e^{\tau^{1/3} t} \tag{18}$$

where τ is the mean enstrophy dissipation rate.

Balloon measurements in the atmosphere (Morel and Larcheveque [17] and Er-El and Peskin [18]) provided evidence comprising second-order structure functions and relative diffusivities which were approximately proportional to the square of the separation length and separation variances which grew approximately exponentially in time, hence supporting the existence of the enstrophy cascade and the Lin relative dispersion scaling result (18) in 2D. Babiano et al. [19] did numerical calculation of relative dispersion in 2D FDT and found that, if the initial pair separation is larger than the energy-injection scale the relative dispersion follows the t^3 law (6) up to the most energetic scales. On the other hand, the relative particle dispersion was found to follow the $e^{\tau^{1/3} t}$ law (18) only in a very short transient stage when the initial separation lies at the bottom of the enstrophy cascade. This was also confirmed by Kowaleski and Peskin [14] via numerical calculations. Particle-path date from a float experiment (Ollitrault et al. [20]) showed that the relative diffusivity varies as R^2 and $R^{4/3}$ for distances smaller and larger, respectively, than a forcing scale of the order of the Rossby radius of deformation. Two particles initially separated between 40 km and 300 km dispersed according to the t^3 law (6) (with relative diffusivity $\sim R^{4/3}$) while those with smaller initial separation distances dispersed according to the $e^{\tau^{1/3} t}$ law (18) (with relative diffusivity $\sim R^2$). The scaling law (18) was confirmed at early times in a laboratory experiment by Jullien [9], but the range of pair separation scales over which this was observed was again very small, so this observation was rather difficult. This situation appears to be traceable to spatial intermittency effects in the enstropy cascade[7], as shown in the following. It is of interest to note that even the numerical investigations of Elhmaidi et al. [30] and Zouari and Babiano [31] showed that relative dispersion in 2D energy cascade is affected by coherent structures and indeed exhibits steeper than the t^3 law, symptomatic of spatial intermittency corrections.

4.1 Inverse Energy Cascade

The spatial intermittency effects in the inverse energy cascade may again be incorporated via the fractal aspects of active turbulence regions. Thus, using the result (Frisch et al. [12]),

$$\alpha = \frac{3 - D_0}{3} \tag{19}$$

(5) becomes,

$$\left\langle [R(t)]^2 \right\rangle \sim t^{\left[3 + \frac{3}{D_0}(2 - D_0)\right]}. \tag{20}$$

Noting that $D_0 < 2$, (20) shows that the effect of spatial intermittency in the energy cascade is to make the relative particle dispersion go steeeper than the t^3 law, in agreement with the

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[7]Direct numerical simulations of freely decaying 2D FDT (McWilliams [21], Benzi et al. [22], Brachet et al. [23], Kida [24], Ohkitani [25], Schneider and Farge [26]) and forced-dissipative 2D FDT (Basdeveant et al. [27], Legras et al. [28], Tsang et al. [29]) showed spatial intermittency in the enstrophy dissipation field caused by the presence of coherent structures.

numerical calculations of Elhmaidi et al. [30] and Zouari and Babiano [31][8].

4.2 Enstrophy Cascade

On incorporating the spatial intermittency effects in the enstrophy cascade as per the homogeneous fractal model for the enstrophy dissipative structures, we have (Shivamoggi [32]),

$$\alpha = \frac{1 + D_0}{3}. \tag{21}$$

Using (21), (5) becomes

$$\left\langle [R(t)]^2 \right\rangle \sim t^{\left[3 + 3\left(\frac{D_0}{2 - D_0}\right)\right]}. \tag{22}$$

The effect of spatial intermittency is to make the relative particle dispersion go slower than the exponential growth law (18) which becomes operational in the space-filling limit $D_0 \Rightarrow 2$. On the other hand, this also shows that, in the presence of spatial intermittency, however small, the exponential growth law (18) is replaced by the power law growth (22). This appears to support the difficulty in observing the exponential growth regime in both laboratory experiments and numerical simulations.

5. Quasi-geostrophic Relative Dispersion

The dynamics of a 3D rapidly rotating fluid is characterized by the geostrophic balance between the Coriolis force and pressure gradient transverse to the axis of rotation. Quasi-geostrophic dynamics refers to the nonlinear dynamics governed by the first-order departure from this linear balance and is inherently 3D. The governing equation is the quasi-geostrophic potential vorticity equation (Charney [33]) for an equivalent barotropic fluid in the f-plane (f being the local Coriolis parameter). The term representing baroclinic effects in the flow in this equation introduces a characteristic length scale, namely the Rossby radius $R_0 \equiv \sqrt{gH}/f$, into the problem (H being the depth of the ocean taken to be uniform and g being the acceleration due to gravity). Consequently, this problem exhibits some interesting departures from the properties of classical 2D turbulence (Shivamoggi [32]).

5.1 Energy Cascade

Upon incorporating the spatial intermittency effects in the energy cascade as per the homogeneous fractal model for the energy dissipative structures, we have (Shivamoggi [32])

$$\alpha = \begin{cases} \frac{4 - D_0}{3} \ , \ \ell_n \gg R_0 \\ \\ \frac{3 - D_0}{3} \ , \ \ell_n \ll R_0. \end{cases} \tag{23a, b}$$

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[8]Indeed, relative particle disperison in the inverse energy cascade was found (Babiano et al. [19], Jullien et al. [16]), unlike Brownian motion, not to be a progressive process, but rather involving sequences of quiet periods and sudden bursts.

Using (23), (5) becomes

$$\langle [R(t)]^2 \rangle \sim \begin{cases} t^{\left(\frac{6}{D_0-1}\right)} , & \ell_n \gg R_0 \\ t^{\left(\frac{6}{D_0}\right)} , & \ell_n \ll R_0. \end{cases} \tag{24a, b}$$

The effect of spatial intermittency in the energy cascade is again to make the relative particle dispersion go steeper than the inertial-range scaling laws,

$$\langle [R(t)]^2 \rangle \sim \begin{cases} t^6 , & \ell_n \gg R_0 \\ t^3 , & \ell_n \ll R_0. \end{cases} \tag{25a, b}$$

Observe that the relative particle dispersion in the baroclinic regime $(\ell_n \gg R_0)$ is greatly increased due to enhanced vortex stretching in this regime.

Further insight into the unusual aspects quasi-geostrophic relative dispersion problem in the inverse energy cascade may be obtained by considering the eddy viscosity development for quasi-geostrophic turbulence.

5.2 Eddy Viscosity for the Inverse Energy Cascade

Kraichnan [34] proposed that the spontaneous development and net energy gain of large-scale structures in the inverse energy cascade from small-scale turbulence can be described by a negative eddy viscosity ν_T (which is either introduced phenomenalogically or derived via closure approximations)[9]. Following Kraichnan [34], one may treat the eddy viscosity as constant and calculate it by balancing the net eddy-viscous *gain* with the energy flux rate ϵ *into* the explicit scales,

$$\int_0^{k_c} 2k^2 \nu_T E(k) dk = -\epsilon \tag{26}$$

k_c being the cut-off wave number so the explicit scales are given by $k < k_c$. The negative sign on the right in (26) arises from the fact that energy is flowing toward smaller wavenumbers (rather than larger wavenumbers).

Using the energy spectra (Shivamoggi [32]),

$$E(k) \sim \begin{cases} c_k \epsilon^{2/3} k^{-5/3} , & kR_0 \gg 1 \\ c_k \epsilon^{2/3} R_0^{-2/3} k^{-7/3} , & kR_0 \ll 1 \end{cases} \tag{27a, b}$$

c_k being a constant.

Using (27), the energy balance relation (26) gives

$$\nu_T \sim \begin{cases} -\frac{2}{3} c_k^{-1} \epsilon^{1/3} k_c^{-4/3} , & k_c R_0 \gg 1 \\ -\frac{1}{3} c_k^{-1} \epsilon^{1/3} (k_c R_0)^{2/3} k_c^{-4/3} , & k_c R_0 \ll 1 \end{cases} \tag{28a, b}$$

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[9]It should be mentioned that the concept of eddy viscosity is not on strong grounds because the basic underlying idea that scales of motion of given size are acted on by smaller scales as if the latter were an augmentation of the equilibrium thermal agitation is not totally valid, thanks to the lack of clear separation of the two scale sizes.

which may be rewritten in the Leslie-Quarini [35] universal form,

$$\nu_T \sim \begin{cases} -\frac{2}{3}c_k{}^{-3/2}\sqrt{\frac{E(k_c)}{k_c}} \ , \ k_c R_0 \gg 1 \\ -\frac{2}{3}\alpha c_k{}^{-3/2}\sqrt{\frac{E(k_c)}{k_c}} \ , \ k_c R_0 \ll 1 \end{cases} \quad (29a, b)$$

(29a) corresponds to the result for classical 2D turbulence given by Kraichnan [34] while (29b) corresponds to the baroclinic regime - observe the explicit appearance, as one would expect, of the baroclinic parameter

$$\alpha \equiv k_c R_0 \quad (30)$$

in the baroclinic regime[10]. (29) shows that the Leslie-Quarini for the eddy viscosity has a certain robustness to it (as also indicated previously by Shivamoggi and Hussaini [37]). Observe further that the higher turbulent transport[11] in the baroclinic regime indicated by (28b) and (29b) is consistent with an enhanced relative particle dispersion indicated by (25a).

For some insight into an actual physical mechanism underlying the negative eddy viscosity, see Appendix.

5.3 Enstrophy Cascade

Upon incorporating the spatial intermittency effects in the enstrophy cascade as per the homogeneous fractal model for the enstrophy dissipative structures we have (Shivamoggi [32])

$$\alpha = \begin{cases} \frac{2+D_0}{3} \ , \ \ell_n \gg R_0 \\ \\ \frac{1+D_0}{3} \ , \ \ell_n \ll R_0. \end{cases} \quad (31a, b)$$

Using (31), (5) becomes

$$\langle [R(t)]^2 \rangle \sim \begin{cases} t^{\left(\frac{6}{1-D_0}\right)} \ , \ \ell_n \gg R_0 \\ t^{\left(\frac{6}{2-D_0}\right)} \ , \ \ell_n \ll R_0. \end{cases} \quad (32a, b)$$

The effect of spatial intermittency in the enstrophy cascade is again to make the relative particle dispersion go slower than the inertial-range scaling laws,

$$\langle [R(t)]^2 \rangle \sim \begin{cases} t^{-6} \ , \ \ell_n \gg R_0 \\ e^{\tau^{1/3}t} \ , \ \ell_n \ll R_0. \end{cases} \quad (33a, b)$$

Observe the particle clumping indicated in the baroclinic regime $(\ell_n \gg R_0)$ because the divorticity sheets favored in the enstrophy cascade baroclinic regime are more likely to occur near vortex nulls (Shivamoggi [38]) and particle clumping is further favored to occur near

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[10]A similar result occurs for eddy viscosity in a compressible turbulence (Shivamoggi and Hussaini [37]) showing the explicit appearance of the Zakharov-Sagdeev [36] compressibility parameter.
[11]This may be seen by rewriting (28) in the following form

$$\nu_T \sim \begin{cases} -\frac{2}{3}c_k{}^{-1}\epsilon^{1/3}R_0{}^{4/3}\alpha^{-4/3} \ , \ \alpha \gg 1 \\ -\frac{2}{3}c_k{}^{-1}\epsilon^{1/3}R_0{}^{4/3}\alpha^{-2/3} \ , \ \alpha \ll 1. \end{cases}$$

vortex nulls (see Section 6). On the other hand, Charney [33] showed that, for $\ell_n \ll R_0$, a potential enstrophy inertial range exists, in which the relative particle dispersion grows exponentially, as indicated by (33b).

6. Compressibility Effects on Relative Particle Dispersion

Intuitively, fluid compressibility is believed to lead to trapping of particles for long times and counteracting their tendency to drift away from each other - strong fluid compressibility would lead to particle clustering (Falkovich et al. [40]). Laboratory experiments and numerical simulations in full-fledged 3D compressible FDT are not at hand yet. Shallow fluid layer flows provide an interesting alternative in this regard because the horizontal divergence of the free-surface flow on a shallow fluid layer is non-zero even though the fluid is incompressible. Consequently, the motion of passive tracer particles[12] (used as surface markers in oceans) is not representative of 2D incompressible turbulence because such particles will respond only to the fluid flow on the free surface and not to the flow normal to the surface. Hence, they sample the horizontal components of the flow-velocity field and provide a convenient framework to analyze the compressibility effects on relative particle dispersion even though the flow velocity is very small compared with the speed of sound (Sommerer and Ott [41]). Laboratory experiments and numerical simulations have therefore investigated relative particle dispersion on a free surface (Cressman et al. [10]). On the other hand, Cressman et al. [10] expressed the necessity to have a theoretical framework to explain their results. We will now proceed to provide one such theoretical formulation.

On assuming barotropic fluid and adiabatic flow processes, we obtain (Shivamoggi [42])

$$\alpha = \frac{\gamma - 1}{3\gamma - 1} \tag{34}$$

where the polytrope exponent $\gamma \, (1 < \gamma < \infty)$ may be treated as a compressibility parameter (the limit $\gamma \Rightarrow \infty$ corresponds to the incompressible fluid limit).

Using (34), (5) becomes

$$\left\langle [R(t)]^2 \right\rangle \sim t^{(3-1/\gamma)}. \tag{35}$$

(35) shows that the effect of compressibility is to make the relative particle dispersion exhibit a power law growth with a scaling exponent smaller than 3 (given by (6)), in agreement with the laboratory experiments and numerical simulations (Cressman et al. [10]).

On the other hand, on incorporating the spatial intermittency effects in compressible turbulence (as indicated by the numerical simulations of Lele et al. [43] Passot et al. [44]), and using a homogeneous fractal model for the kinetic energy dissipative structures, we have (Shivamoggi [45]),

$$\alpha = \left(\frac{\gamma - 1}{3\gamma - 1} \right) (D_0 - 2). \tag{36}$$

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[12]Particle tracers which have the same density as that of the carrier fluid and very small size can be approximated as point-like particles having the same velocity as that of the carrier fluid at the position of the particle.

Using (36), (5) becomes

$$\left\langle [R(t)]^2 \right\rangle \sim t^{\left[\frac{3-1/\gamma}{1+\left(\frac{\gamma-1}{2\gamma}\right)(3-D_0)} \right]}. \tag{37}$$

Noting that $D_0 < 3$, (37) shows that the effect of spatial intermittency is to cause further reduction in the relative particle dispersion. On the other hand, on noting that the dissipative structures in a compressible turbulence are typically shock-wave like ($D_0 = 2$), (37) becomes

$$\left\langle [R(t)]^2 \right\rangle \sim t^2 \, , \forall \gamma \tag{38}$$

indicating that relative particle dispersion in intermittent compressible turbulence occurs in the ballistic regime. This appears to be physically plausible because, in the presence of shock waves, particle clustering[13] renders the velocity increment become independent of the particle separation, and hence forcing the relative particle dispersion go ballistic. It is of interest to note, in comparison with (38), that laboratory experiments (Cressman et al. [10]) indicated a scaling exponent of 1.65 while numerical simulations (Cressman et al. [10]) indicated a scaling exponent of 1.80. Further, (37) also shows that compressibility effects, no matter how strong, cannot change the super-diffusive nature of relative particle dispersion. Indeed, in the infinite compressibility limit $\gamma \Rightarrow 1$, (37) yields

$$\left\langle [R(t)]^2 \right\rangle \sim t^2 \, , \ \forall D_0 \tag{39}$$

pertaining again to the ballistic regime (38)!

7. Discussion

In this paper, several aspects of the turbulent relative particle dispersion are considered, many of which are motivated by laboratory experiment and numerical simulation results. These are,

* spatial intermittency effects,

    (a) reduction of relative particle dispersion in 3D, corroborating the numerical simulation results (Boffetta and Sokolov [7]);

    (b) prevalence of power-law scaling of relative particle dispersion in 2D enstrophy cascade (no matter how weak spatial intermittency effects are), corroborating the difficulty in observing Lin [8] scaling law in laboratory experiments (Jullien [9]) and numerical simulations (Babiano et al. [19] and Kowaleski and Peskin [14]);

* quasi-geostrophic aspects exhibiting an enhanced relative particle dispersion in the baroclinic regime and some insight into this aspect has been explored via a negative eddy viscosity development;

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[13]Thanks to a compressible-fluid flow like flow situation prevailing on the free-surface of shallow fluid layers, local convergence and divergence regions of particle density are observed there (Elhmaidi et al. [30] and Cielsik et al. [46]). In convergence regions where particles tend to clump there is downwelling and vice versa. Further, downwellings are found to occur near strain-dominated (thin elongated) regions while upwellings are found to occur near rotation-dominated (patch like) regions.

* reduction of relative particle dispersion and prevalence of the ballistic regime due to compressibility effects, corroborating the laboratory experiment and numerical simulation results (Cressman et al. [10]) where compressibility is effectively produced on the free surface of a shallow fluid layer.

It should be mentioned, however, that the 3D RO scaling result (3) has received little experimental support due to the difficulty of performing Lagrangian measurements over a broad enough range of time and with sufficient accuracy. Even in recent laboratory experiments (Ott and Mann [47], Sawford [48], Bourgoin et al. [49], Salazar and Collins [50]) with high-speed photography to track particles, and in numerical simulations (Yeung [51]) with the highest possible resolution possible for homogeneous isotropic turbulence, it is known to be hard to obtain an extended range with the RO scaling. The difficulty appears to be due to,

* contamination of the inertial range by dissipative effects at the ultraviolet end and the external forcing effects at the infrared end of the spectrum caused by inadequate scale separation;

* persistent memory of initial separation.

Acknowledgments

Appendix. Physical Phenomenological Derivation of Negative Eddy Viscosity

A sound phenomenological demonstration of negative eddy viscosity via a simple model that captures the essential physics is apparently not at hand despite Kraichnan's [34] attempts motivated by complicated analytic developments. Analytical approaches used by Sivashinsky et al. [52], [53] and Gama et al. [54] are restricted to flows possessing special symmetries. We propose to contribute toward this quest and give a simple phenomenological derivation of negative eddy viscosity for 2D energy cascade.

We borrow from a basic idea for a physical mechanism underlying the negative eddy viscosity propsed by Kraichnan [34] - the large scales strain the small scales while a secondary flow associated with small scales grows. Further, the deterministic large-scale flow and a random small-scale vorticity field are characterized by disparate scales (Tur et al.[55] and

Fidutenko [56]). Consider therefore a random homogeneous small-scale vorticity field ω superposed on a stationary large-scale flow with stream function Ψ, with velocity given by

$$\mathbf{V} = \hat{\mathbf{i}}_z \times \nabla \psi. \tag{A·1}$$

Linearizing about the large-scale flow (in analogy with a phenomenological development for MHD turbulence sketched by Pouquet [57]) we obtain, for the small-scale flow, with velocity $\mathbf{v}$,

$$\frac{\partial \mathbf{v}}{\partial t} = \omega \nabla \Psi \tag{A·2}$$

where,

$$\boldsymbol{\omega} \equiv \nabla \times \mathbf{v} = \omega \hat{\mathbf{i}}_z.$$

Assuming a normal-mode type evolution in time for the small-scale flow,

$$\mathbf{v} \sim e^{\sigma t} \tag{A·3}$$

so $1/\sigma$ may be interpreted as a coherence time of the small-scale flow, equation (A·2) becomes

$$\mathbf{v} = \frac{\omega}{\sigma} \nabla \Psi. \tag{A·4}$$

On the other hand, the large-scale vorticity Ω evolves according to

$$\frac{\partial}{\partial t} \langle \Omega \rangle + \nabla \cdot \langle \mathbf{v} \omega \rangle = 0. \tag{A·5}$$

Assuming the large-scale flow to comply with the Beltrami condition[14],

$$\langle \Omega \rangle = L^2 \langle \Psi \rangle \tag{A·6}$$

L being a constant (of dimension length), and using equation (A·4), equation (A·5) becomes,

$$\frac{\partial}{\partial t} \langle \Psi \rangle = -\frac{L^2}{\sigma} \langle \omega^2 \rangle \nabla^2 \Psi \tag{A·7}$$

which implies that the nonlinear growth rate of the flow has the form of negative eddy viscosity given by,

$$\nu_T \sim -\frac{L^2}{\sigma} \langle \omega^2 \rangle. \tag{A·8}$$

Noting the spectral behaviors,

$$\left.\begin{array}{l} \langle \omega^2 \rangle \sim \epsilon^{2/3} k^{-2/3} \\ \sigma \sim \epsilon^{1/3} k^{2/3} \end{array}\right\} \tag{A·9}$$

(A·8) becomes

$$\nu_T \sim \epsilon^{1/3} k^{-4/3} \tag{A·10}$$

in agreement with (28a)!

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[14] A similar assumption was also made in the negative-viscosity development of Sivashinsky et al. [52], [53].

On the other hand, on recognizing that turbulent transport in the inverse cascade is actually a competition between flow advection and vortex coalescence, (A·8) may be rewritten as

$$\nu_T \sim \frac{1}{\sigma} \left(\langle u^2 \rangle - L^2 \langle \omega^2 \rangle \right). \tag{A·11}$$

A phenomenological derivation of the first term in (A·11) may be given as follows. Consider a flow velocity $\mathbf{v} = \langle u, v \rangle$ with a displacement vector $\boldsymbol{\ell} = \langle \ell_1, \ell_2 \rangle$. If σ^{-1} is a typical time scale, we have on complying with the continuity equation (*á la* Batchelor [58] and Kraichnan [59])

$$\left. \begin{array}{l} u \sim d\ell_1/dt \sim \sigma \ell_1 \\ v \sim d\ell_2/dt \sim -\sigma \ell_2 \end{array} \right\} \tag{A·12a, b}$$

while the vorticity is given by

$$\omega \sim \frac{v}{\ell_1} - \frac{u}{\ell_2} \sim -u\ell_2 \left(\frac{1}{\ell_1^2} + \frac{1}{\ell_2^2} \right). \tag{A·13}$$

The eddy viscosity (or turbulent diffusivity) is given by

$$\nu_T \sim \frac{1}{2} \frac{d}{dt} \langle \ell^2 \rangle \sim \sigma \left(\langle \ell_1^2 \rangle - \langle \ell_2^2 \rangle \right). \tag{A·14}$$

Using (A·12) and (A·13), (A·14) becomes

$$\nu_T \sim \frac{1}{\sigma} \left(\langle u^2 \rangle - L^2 \langle \omega^2 \rangle \right) \tag{A·15}$$

in agreement with (A·11). Here,

$$L^2 \equiv \frac{\langle \ell_1^2 \rangle}{\left(1 + \langle \ell_1^2 \rangle / \langle \ell_2^2 \rangle \right)^2}. \tag{A·16}$$

Observe that, according to (A·11) or (A·15), the negative eddy-viscosity effect disappears if the small-scale flow decorrelates rapidly in time (i.e., $\sigma \Rightarrow \infty$), as Kraichnan [34] pointed out.

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